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NUMERICAL STUDY OF THE STOKES EQUATIONS

S. D. Algazin

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The exterior problem for linearized stationary Navier–Stokes equations (Stokes equations) for flow around an axisymmetric body at small Reynolds numbers is considered. No assumptions are made as to the direction of the velocity vector in an undisturbed flow. Thus, the problem is three-dimensional in the general case. A numerical study of these equations showed that they are poorly conditioned. A numerical algorithm for solving poorly conditioned Stokes equations is suggested. The algorithm has no saturation, i.e., the smoother the unknown solution, the higher the accuracy.

1. Formulation of the Problem and Choice of Coordinate System. In Cartesian coordinates (x_1, x_2, x_3), the system of Stokes equations has the form

$$\frac{\partial p}{\partial x_i} = \frac{1}{\mathrm{Re}}\Delta v^i, \quad i = 1,2,3; \tag{1.1}$$

$$\frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2} + \frac{\partial v^3}{\partial x_3} = 0, \tag{1.2}$$

where Re is the Reynolds number, (v^1, v^2, v^3) is the velocity vector, and p is pressure. Both dependent and independent variables that enter Eqs. (1.1) and (1.2) are nondimensionalized by the standard procedure. The typical linear dimension L_a and the length of the flow velocity vector v_∞ at infinity are taken as characteristic quantities. Then, for example, $p = (P - p_\infty)/(\rho v_\infty^2)$, where P is dimensional pressure, ρ is the density of the fluid, and p_∞ is the pressure in the undisturbed flow (at infinity). Thus, to determine the flow parameters, the velocity vector (v^1, v^2, v^3), and the pressure p, one has to find a solution of the system of equations (1.1) and (1.2) subject to the following boundary conditions:

$$v^i|_{\partial\Omega} = 0, \quad i = 1,2,3, \quad v^i|_\infty = v^i_\infty, \quad i = 1,2,3, \quad p|_\infty = 0.$$

Here, Ω is the body in question, axisymmetric about the x_3 axis, $\partial\Omega$ is its boundary, and v^i_∞ $(i = 1,2,3)$ is the liquid velocity in the free stream (at infinity).

As a consequence of Eqs. (1.1) and (1.2), we have

$$\Delta p = 0, \tag{1.3}$$

that is, the pressure is a harmonic function outside the axisymmetric body, a circumstance used below.

We introduce a system of curvilinear coordinates (r, ϑ, φ) connected with the Cartesian coordinates (x_1, x_2, x_3) through the relations [1]

$$x_1 = v(r,\vartheta)\cos\varphi, \quad x_2 = v(r,\vartheta)\sin\varphi, \quad x_3 = u(r,\vartheta). \tag{1.4}$$

We denote by G the region obtained by passing a meridional section through the body Ω and choose the functions u and v in the following manner. Let $\psi(z) = u(r,\vartheta) + iv(r,\vartheta)$ and $z = r\exp(i\vartheta)$ be a conformal mapping of the circle $|z| = r \leqslant 1$ onto the exterior of the region G, with the center of the circle going into an infinitely distant point. It is convenient to regard (r, ϑ, φ) as spherical coordinates, since in this case relations (1.4) define a mapping of the unit ball onto the exterior of the body Ω.

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For an ellipsoid of revolution about the x_3 axis,

$$\frac{x_1^2}{b^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{a^2} = 1 \tag{1.5}$$

an analytical form for the functions u and v is known [2]. Relation (1.4) maps the surface of the unit ball onto the surface of Ω. The boundary conditions set on $\partial\Omega$ are then transferred to the ball surface. The boundary conditions set at infinity are transferred to the ball center.

Usually, when a curvilinear coordinate system is used, equations for vector quantities are written in terms of their projections onto the axes of their own basis, whose coordinate vectors are directed tangentially to the coordinates lines. This basis is dependent on the coordinates of the point in space. An approach like this is inconvenient in the present case because on the x_3 axis relation (1.4) is no longer one-to-one (if $v = 0$, φ is arbitrary). This causes the appearance of singularities in the solution, which are conditioned by the "poor" choice of the coordinate system rather than by the essence of the problem. Note that the spherical coordinate system has a similar "shortcoming."

The way out of this situation is as follows: we retain the projections of the velocity vector v^i ($i = 1, 2, 3$) onto the axes of the Cartesian coordinate system as sought-for functions but replace the independent variables x_1, x_2, and x_3 with r, ϑ, and φ by substituting (1.4). We then get

$$\alpha \cos\varphi \frac{\partial p}{\partial r} + \beta \cos\varphi \frac{\partial p}{\partial \vartheta} - \frac{1}{v} \sin\varphi \frac{\partial p}{\partial \varphi} = \frac{1}{\mathrm{Re}} (\Delta V^1 + f_1); \tag{1.6}$$

$$\alpha \sin\varphi \frac{\partial p}{\partial r} + \beta \sin\varphi \frac{\partial p}{\partial \vartheta} + \frac{1}{v} \cos\varphi \frac{\partial p}{\partial \varphi} = \frac{1}{\mathrm{Re}} (\Delta V^2 + f_2); \tag{1.7}$$

$$\frac{r v_\vartheta}{w^2} \frac{\partial p}{\partial r} - \frac{r v_r}{w^2} \frac{\partial p}{\partial \vartheta} = \frac{1}{\mathrm{Re}} (\Delta V^3 + f_3); \tag{1.8}$$

$$\alpha \cos\varphi \frac{\partial V^1}{\partial r} + \beta \cos\varphi \frac{\partial V^1}{\partial \vartheta} - \frac{1}{v} \sin\varphi \frac{\partial V^1}{\partial \varphi} + \alpha \sin\varphi \frac{\partial V^2}{\partial r}$$

$$+ \beta \sin\varphi \frac{\partial V^2}{\partial \vartheta} + \frac{1}{v} \cos\varphi \frac{\partial V^2}{\partial \varphi} + \frac{r v_\vartheta}{w^2} \frac{\partial p}{\partial r} - \frac{r v_r}{w^2} \frac{\partial p}{\partial \vartheta} = f_4, \tag{1.9}$$

where

$$\alpha(r, \vartheta) = -r u_\vartheta / w^2 \quad (w^2 = u_\vartheta^2 + v_\vartheta^2); \qquad \beta(r, \vartheta) = (1 + r u_\vartheta v_r / w^2) / v_\vartheta;$$

$$f_i = -r v_\infty^i (1 + r v_r / v) / w^2 \quad (i = 1, 2, 3); \qquad f_4 = v_\infty^1 \alpha \cos\varphi + v_\infty^2 \alpha \sin\varphi + v_\infty^3 r v_\vartheta / w^2;$$

$$v^i = (1 - r) v_\infty^i + V^i \quad (i = 1, 2, 3). \tag{1.10}$$

The replacement of the unknown functions v^i by V^i ($i = 1, 2, 3$) in accordance with formula (1.0) is carried out in order that the boundary conditions for the velocity be uniform.

$$V^i|_{r=0} = V^i|_{r=1} = 0, \quad i = 1, 2, 3. \tag{1.11}$$

This is required to facilitate discretization of the Laplacian. For pressure, we have the boundary condition

$$p|_{r=0} = 0. \tag{1.12}$$

The Laplacian of the functions V^i ($i = 1, 2, 3$) in terms of the variables (r, ϑ, φ) takes the form

$$\Delta V^i = \frac{r}{v w^2} \left(\frac{\partial}{\partial r} \left(r v \frac{\partial V^i}{\partial r} \right) \right) + \frac{\partial}{\partial \vartheta} \left(\frac{v}{r} \frac{\partial V^i}{\partial \vartheta} \right) + \frac{1}{v^2} \frac{\partial^2 V^i}{\partial \varphi^2}. \tag{1.13}$$

Thus, Eqs. (1.6)–(1.9) with boundary conditions (1.11) and (1.12) have to be solved in the unit ball.

2. Discrete Laplacian and Discrete Stokes Equations. We discretize the Laplacian (1.13) with the uniform boundary conditions (1.11) by using the procedure in [3].

Thus, we get a discrete Laplacian in the form of an h-matrix

$$H = \frac{2}{L} \sum_{k=0}^{l}{}' \Lambda_k \otimes h_k, \quad L = 2l + 1. \tag{2.1}$$

Here, the prime indicates that the term with $k = 0$ is taken with the coefficient $1/2$; the symbol $\otimes$ designates the Kronecker product of the matrices, and h is a matrix of order $L \times L$ with the elements

$$h_{kij} = \cos k \frac{2\pi(i-j)}{L} \quad (i, j = 1, 2, \ldots, L);$$

Λ_k is the matrix of the discrete operator corresponding to the differential operator

$$\frac{r}{vw^2}\left(\frac{\partial}{\partial r}\left(rv\frac{\partial \Phi}{\partial r}\right)\right) + \frac{\partial}{\partial \vartheta}\left(\frac{v}{r}\frac{\partial \Phi}{\partial \vartheta}\right) - \frac{k^2}{v^2}\frac{\partial^2 \Phi}{\partial \varphi^2}, \quad k = 0, \ldots, l \tag{2.2}$$

with the boundary conditions

$$\Phi\,|_{r=0} = \Phi\,|_{r=1} = 0. \tag{2.3}$$

To discretize differential operator (2.2), (2.3) we take a mesh with n nodes for ϑ

$$\vartheta_\nu = \frac{\pi}{2}(y_\nu + 1), \quad y_\nu = \cos\varepsilon_\nu, \quad \varepsilon_\nu = \frac{(2\nu-1)\pi}{2n}, \quad \nu = 1, 2, \ldots, n,$$

and apply the interpolation formula

$$g(\vartheta) = \sum_{\nu=1}^{n} \frac{T_n(y) g_\nu}{n \dfrac{(-1)^{\nu-1}}{\sin\varepsilon_\nu}(y - y_\nu)}, \quad y = (2\vartheta - \pi)/\pi, \tag{2.4}$$

where $g_\nu = g(\vartheta_\nu)$ $(\nu = 1, 2, \ldots, n)$; $T_n(y) = \cos(n \arccos y)$.

Differentiation of the interpolation formula (2.4) gives the first and second derivatives with respect to ϑ in relations (2.2).

We choose a mesh with m nodes along r

$$r_\nu = (1 + z_\nu)/2, \quad z_\nu = \cos\chi_\nu, \quad \chi_\nu = (2\nu - 1)\pi/(2m), \quad \nu = 1, 2, \ldots, m,$$

and employ the interpolation formula

$$q(r) = \sum_{\nu=1}^{m} \frac{T_m(z)(r-1) r q_k}{m \dfrac{(-1)^{\nu-1}}{\sin\chi_\nu}(r_\nu - 1) r_\nu (z - z_\nu)}, \quad q_\nu = q(r_\nu), \quad z = 2r - 1. \tag{2.5}$$

Differentiation of the interpolation formula (2.5) gives the first and the second derivatives with respect to r, in expression (2.2). By differentiating interpolation formulas (2.4) and (2.5), we obtain the values of derivatives with respect to ϑ and r that enter the left-hand side of the continuity equation (1.9).

To discretize the derivatives of pressure with respect to r, we use the interpolation formula

$$q(r) = \sum_{\nu=1}^{m} \frac{T_m(z)\, r\, q_k}{m \dfrac{(-1)^{\nu-1}}{\sin\chi_\nu} r_\nu (z - z_\nu)}. \tag{2.6}$$

The quantities entering formula (2.6) are determined above. The values of the first derivatives of pressure with respect to r on the left-hand side of relations (1.6)–(1.8) are obtained by differentiating the interpolation formula (2.6).

To derive a formula for numerical differentiation with respect to φ, we consider the interpolation formula

$$s(\varphi)=\frac{2}{L}\sum_{k=0}^{2l}D_l(\varphi-\varphi_k)s_k,\quad L=2l+1, \tag{2.7}$$

where

$$s_k=s(\varphi_k);\quad \varphi_k=2\pi k/L\quad (k=0,1,\dots,2l);\quad D_l(\varphi-\varphi_k)=0.5+\sum_{j=1}^{l}\cos j(\varphi-\varphi_k).$$

We determine the values of derivatives with respect to φ by differentiating formula (2.7).

To obtain discrete Stokes equations, one should replace the derivatives in Eqs. (1.6)–(1.9) by discrete derivatives found by differentiating the corresponding interpolation formulas (2.4)–(2.7). The Laplacian is replaced by the matrix H. In place of the functions V^1, V^2, V^3, and p, the discrete Stokes equations contain their values in the nodes of the mesh $(\vartheta_\nu, r_\mu, \varphi_k)$, $\nu=1,2,\dots,n$, $\mu=1,2,\dots,m$, and $k=0,1,2,\dots,2l$. As a result, we have a system of $4mnL$ linear equations. The system of discrete equations cannot be written in explicit form because of its awkwardness. For instance, for $m=n=10$ and $L=9$, the system has the order of 3600.

To study the condition number of this system of linear equations, the eigenvalues of the Laplace operator with boundary conditions (1.11) were calculated. To do this, it is sufficient to calculate the eigenvalues of the matrices Λ_k, $k=0,1,\dots,l$ [4]. Numerical experiments have shown that the eigenvalues of the Laplace operator have two condensation points, 0 and $-\infty$. Thus, the values of the norms of the matrices H and H^{-1} are large. They grow as the number of nodes increases. This is what distinguishes exterior problems from interior ones.

The matrix of the discrete Stokes equations is of block form

$$A=\begin{Vmatrix} H & 0 & 0 & P_1\\ 0 & H & 0 & P_2\\ 0 & 0 & H & P_3\\ u_1 & u_2 & u_3 & 0\end{Vmatrix},$$

where H is a discrete Laplacian, P_i $(i=1,2,3)$ are matrices obtained after discretizing terms with pressure, u_i $(i=1,2,3)$ are matrices obtained after discretizing the discontinuity equation. All these matrices are of order $R\times R$ ($R=mnl$ is the number of nodes). We denote

$$A_{n-1}=\begin{Vmatrix} H & 0 & 0\\ 0 & H & 0\\ 0 & 0 & H\end{Vmatrix},\quad v_n=(u_1,u_2,u_3),\quad u_n=(P_1,P_2,P_3)'.$$

The reciprocal of the matrix A is sought in the form

$$A^{-1}=\begin{Vmatrix} P_{n-1} & r_n\\ q_n & \alpha_n^{-1}\end{Vmatrix}.$$

Here, P_{n-1} is a matrix of order $3R\times 3R$, $q_n=(q_1,q_2,q_3)$ where q_i $(i=1,2,3)$ are matrices of order $R\times R$, $r_n=(r_1,r_2,r_3)'$ where r_i $(i=1,2,3)$ are matrices of order $R\times R$. We then get $q_n=-\alpha_n^{-1}v_nA_{n-1}^{-1}$, $P_{n-1}=A_{n-1}^{-1}+A_{n-1}^{-1}u_n\alpha_n^{-1}v_nA_{n-1}^{-1}$, and $r_n=-A_{n-1}^{-1}u_n\alpha_n^{-1}$ ($\alpha_n=-u_1H^{-1}p_1-u_2H^{-1}p_2-u_3H^{-1}p_3$ is a matrix of order $R\times R$).

Thus, it is easily seen that, because of the properties of the matrices H and H^{-1} described above, the value of the norm of A and A^{-1} is large. This value grows as the number of nodes increases, i.e., the system of discrete Stokes equations is poorly conditioned. This results from the poorly conditioned differential Stokes equations in the unbounded region (i.e., in the exterior of the axisymmetric body) and is caused by the spectrum of the Laplace operator in the region under consideration.

An approximate method for solving poorly conditioned discrete Stokes equations is considered below. We now turn to discussing the properties of the discretization carried out. The classical approach to discretization of the equations of mathematical physics consists in replacing derivatives with finite differences. This approach has an essential shortcoming: it is not sensitive to the smoothness of solution of the problem in question, i.e., the discretization error does not depend on the smoothness of the unknown solution. In other words, the finite-difference algorithms lead to numerical methods with saturation [5]. For this reason, an interpolation of solution with polynomials (algebraic or trigonometric ones) was applied above for discretization of the Stokes equations. The derivatives of the sought-for functions in the Stokes equations were calculated by differentiating the interpolation formulas. This method of discretization has no saturation, because the smoother the unknown function, the more accurately it is approximated by the interpolating polynomial [5]. Such a property of the algorithm makes it possible to carry out calculations on a fairly scarce mesh, when the condition number of the discrete Stokes equations is not very large.

3. Determination of Pressure. It has been pointed out above [see (1.3)] that pressure is a harmonic function. Let us consider a more general eigenvalue problem for the Laplace operator in the unit ball with a deleted center:

$$\Delta p = \lambda p, \quad p\,|_{r=0} = 0\,. \tag{3.1}$$

We are interested in the eigenfunctions of the boundary problem (3.1) that correspond to the zero eigenvalue $\lambda = 0$. The replacement of relation (1.3) by a more general problem (3.1) can be explained by the fact that the methods for solving finite-dimensional eigenvalue problems are very well developed [6]. So are the methods for discretizing a Laplacian [3, 4].

In discrete form, the boundary-value problem (3.1) can be reduced to calculating the eigenvalues of an h-matrix, i.e., to solving the algebraic eigenvalue problem

$$H\mathbf{p} = \lambda\,\mathbf{p} \tag{3.2}$$

($\mathbf{p}$ is a vector with length nmL whose components are the values of the sought-for pressure in the nodes of the mesh). The matrix H is constructed by using formula (2.1). However, the interpolation formula (2.6) subject to the boundary condition [see (3.1)] is employed for numerical differentiation with respect to r. Solving the finite-dimensional problem (3.2), we determine the eigenvalues close to zero. The corresponding eigenvector is determined within a constant factor c. Having substituted the found solution into the discrete Stokes equations, we easily determine the velocity components from the equations of motion. To do so, one should reverse the h-matrix by using the formula in [4]:

$$H^{-1} = \frac{2}{L} \sum_{k=0}^{l}{}' \Lambda_k^{-1} \otimes h_k, \quad L = 2l + 1$$

(the formula can be verified by immediate multiplication) and calculate the product of this matrix and some vectors. It remains now to choose the constant c in such a manner that the continuity equation is satisfied. We substitute the velocity components found above into the continuity equation and obtain a set of $R = mnL$ equations for determining the constant. This is a set of overdefined linear equations that serves for discarding "unnecessary" solutions and finding the constant c. For the desired solution, the constants c, determined from the discrete continuity equation, must necessarily be approximately equal to one another. Any of them or their arithmetic mean can be taken as the value of the constant that we are looking for. For irrelevant solutions, the constants c differ greatly from one another. Such solutions must be discarded.

Note that one can reduce calculations of the eigenvalues and eigenvectors of the h-matrix to calculating those of the matrices Λ_k ($k = 0, 1, \ldots, l$) whose order is smaller [4]. Thus, it is possible to determine all the eigenvalues and eigenvectors of the h-matrix of order 900×900.

4. Results of Numerical Experiments. The numerical experiments were conducted for a ball with $a = b = 1$ and for an ellipsoid of revolution with $a = 1$ and $b = 0.5$ or 0.95 [see (1.5)] on a mesh with 225 ($m = n = 5$ and $L = 9$), 900 ($m = n = 10$ and $L = 9$), and 2025 nodes ($m = n = 15$ and $L = 9$). A flow with parameters $v_\infty^1 = 1$ and $v_\infty^2 = v_\infty^3 = 0$ was taken as the boundary condition for velocity in the free stream. In all the calculations, Re was taken to be 0.01.

Let us first discuss the results of the calculations for the ball. Two close-to-zero eigenvalues of the matrix

Λ_0, $\lambda_{24} = -0.3 \cdot 10^{-5}$ and $\lambda_{25} = -0.7 \cdot 10^{-18}$, were determined on the mesh with 225 nodes ($m = n = 5$ and $L = 9$). The remaining eigenvalues had order between 10^{-2} and 10^2. The only close-to-zero eigenvalue was determined for the matrix Λ_1: $\lambda_{24} = -0.5 \cdot 10^{-5}$. Other eigenvalues were of order 10^{-2}–10^3. The matrices Λ_2, Λ_3, and Λ_4 have eigenvalues of order 10^{-2}–10^3, 10^{-1}–10^4, and 10^{-1}–10^4, respectively. Consequently, they do not have eigenvalues that can be interpreted as close to zero. The second calculation was carried out on a mesh with 900 nodes ($m = n = 10$ and $L = 9$). The matrix Λ_0 has two real close-to-zero eigenvalues: $\lambda_{99} = 0.2 \cdot 10^{-11}$ and $\lambda_{100} = -0.4 \cdot 10^{-18}$. In addition, there is a pair of complex close-to-zero eigenvalues with real parts $\lambda_{97} = \lambda_{98} = -0.5 \cdot 10^{-7}$. Other eigenvalues are of order 10^{-3}–10^3. The matrix Λ_1 has a close-to-zero eigenvalue $\lambda_{100} = -0.2 \cdot 10^{-12}$. In addition, there is a pair of complex close-to-zero eigenvalues with real parts $\lambda_{98} = \lambda_{99} = -0.2 \cdot 10^{-8}$. Other eigenvalues are of order 10^{-4}–10^4. The eigenvalues of matrices Λ_2, Λ_3, and Λ_4 are of order 10^{-6}–10^5, 10^{-4}–10^5 and 10^{-3}–10^5, respectively. Thus, the calculations conducted show that for the unit ball the h-matrix has four families of eigenvectors which yield close-to-zero eigenvalues (note that the eigenvalue of the matrix Λ_1 is twofold [4]).

The calculation of the eigenvectors of the h-matrix for the ball was conducted on a mesh with 900 nodes ($m = n = 10$ and $L = 9$). The four sought-for families of eigenvalues for problem (3.1) in the unit ball can be easily guessed. The eigenvectors of the h-matrix, corresponding to the close-to-zero real eigenvalues of the matrix Λ_0, yield two families of eigenfunctions independent of φ:

$$p_1 = cr \tag{4.1}$$

corresponds to the eigenvalue λ_{100} of the matrix Λ_0, and

$$p_2 = c_1 r \ln((1 - \cos\vartheta)/(1 + \cos\vartheta)) + c_2 r \tag{4.2}$$

corresponds to the eigenvalue λ_{99} of the matrix Λ_0. Speaking more precisely, one of the invariant subspaces of the Laplace operator (3.1), corresponding to the zero eigenvalue, has form (4.2), i.e., is two-dimensional. Calculations give two close-to-zero eigenvalues of the matrix Λ_0 of order 100×100 (λ_{100} and λ_{99}). As has been pointed out above, an eigenfunction of form (4.1) corresponds to the eigenvalue λ_{100} (this fact is confirmed by numerical calculations), and some eigenfunctions of family (4.2) corresponds to the eigenvalue λ_{99}.

The eigenvectors of the h-matrix, corresponding to the real close-to-zero eigenvalue λ_{100} of the matrix Λ_1, yield two families of eigenfunctions dependent on φ

$$p_3 = c_3 r^2 \sin\vartheta \cos\varphi; \tag{4.3}$$

$$p_4 = c_4 r^2 \sin\vartheta \sin\varphi. \tag{4.4}$$

Family (4.3) of eigenfunctions corresponds to the solution in [7] for a ball. Families (4.1), (4.2) and (4.4) yield irrelevant solutions which do not satisfy the continuity equation (see Section 3).

Next, the constant c_3 was calculated from the continuity equations (see Section 3). The arithmetic mean of constants close to one another was taken as the desired constant. These constants were determined from the discrete continuity equation. The value $c_3 = 144.09$ was obtained (the eigenvector of the matrix Λ_1, corresponding to the eigenvalue λ_{100}, was normalized by the maximum of its length). The resulting approximate solution was compared with the exact one [7]. The calculations show that the maximum relative error equals 0.26%.

The second calculation was carried out for an ellipsoid with the semiaxes $a = 1$ and $b = 0.5$. It was found on a mesh with 225 nodes ($m = n = 5$ and $L = 9$) that the matrix Λ_0 has one close-to-zero eigenvalue $\lambda_{25} = -0.3 \cdot 10^{-5}$. Other eigenvalues were of the order of 10^{-2}–10^2. The eigenvalues of the matrices Λ_1, Λ_2, Λ_3 and Λ_4 were of order 10^{-2}–10^3, 10^{-2}–10^4, 10^{-1}–10^4 and 10^{-1}–10^5. Thus, the number of nodes is apparently insufficient. On a mesh with 900 nodes ($m = n = 10$ and $L = 9$), the matrix Λ_0 has two close-to-zero eigenvalues: $\lambda_{99} = 0.4 \cdot 10^{-6}$, $\lambda_{100} = 0.2 \cdot 10^{-9}$. Other eigenvalues are of the order of 10^{-3}–10^4. The matrix Λ_1 has one close-to-zero eigenvalue: $\lambda_{94} = 0.3 \cdot 10^{-5}$. Other eigenvalues are of the order of 10^{-3}–10^4. The matrices Λ_2, Λ_3, and Λ_4 have eigenvalues of order 10^{-3}–10^5, 10^{-3}–10^5, and 10^{-2}–10^6.

Further, calculations on a mesh with 2025 nodes ($m = n = 15$ and $L = 9$) were carried out. The eigenvalues of the matrices Λ_0 and Λ_1 were calculated. The matrix Λ_0 has two close-to-zero eigenvalues:

$\lambda_{224} = -0.1 \cdot 10^{-9}$ and $\lambda_{225} = -0.2 \cdot 10^{-13}$. Other eigenvalues are of the order of 10^{-5}–10^{4}. The Matrix Λ_1 has one close-to-zero eigenvalue $\lambda_{221} = -0.1 \cdot 10^{-8}$. Other eigenvalues are of the order of 10^{-4}–10^{5}. Thus, the h-matrix for the ellipsoid also has four families of eigenvectors corresponding to close-to-zero eigenvalues of the matrices Λ_0 and Λ_1. We are interested in an eigenfunction that is even with respect to φ and corresponds to a close-to-zero eigenvalue of the matrix Λ_1 (a disturbance of the corresponding eigenfunction for the ball). An approximate calculation of this eigenfunction was conducted on a mesh with 900 nodes ($m = n = 10$ and $L = 9$).

The calculation results show that the constants c_i ($i = 1, 2, \ldots, 900$) differ from one another rather significantly, their mean value being 318.31. It is obvious that 900 nodes are insufficient to find this eigenfunction (we should not forget that the eigenvalue of the matrix Λ_1 corresponding to the desired eigenvector has order 10^{-5}, i.e., is not sufficienly close to zero). To check this hypothesis, calculations for an ellipsoid with the semiaxes $a = 1$ and $b = 0.95$ were carried out on the mesh with 900 nodes. The eigenvalues of the matrices Λ_0 and Λ_1 were calculated. The matrix Λ_0 turned out to have two close-to-zero eigenvalues: $\lambda_{99} = 0.2 \cdot 10^{-11}$ and $\lambda_{100} = 0.1 \cdot 10^{-16}$. In addition, there is a pair of complex close-to-zero eigenvalues with real parts $\lambda_{97} = \lambda_{98} = -0.6 \cdot 10^{-7}$. Other eigenvalues are of the order of 10^{-3}–10^{3}. The matrix Λ_1 has one real close-to-zero eigenvalue $\lambda_{100} = 0.2 \cdot 10^{-12}$ and a pair of complex eigenvalues with real parts $\lambda_{98} = \lambda_{99} = -0.2 \cdot 10^{-8}$. Calculation of the eigenvector was conducted for the eigenvalue λ_{100} of the matrix Λ_1. The spread of c_i's ($i = 1, 2, \ldots, 900$) was between 147.85 and 160.57, the mean value being $c = 152.36$. The maximum relative error between the found solution and the solution in the ball was 6%. Thus, to employ this approximate solution of the discrete Stokes equations, one should use a calculation mesh such that the close-to-zero eigenvalues of h-matrix (3.2) are of the order of 10^{-12}.

An AT-386 PC with 640 K of RAM operating at 25 MHz was used for the calculations. As is seen from the calculations described above, one can employ numerical methods for studying the flow of a viscous incompressible liquid around bodies close to a ball at small Reynolds numbers. A more powerful computer is required to study flows around bodies of complex shape.

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